

NON-DIAGONAL METRIC ON A PRODUCT RIEMANNIAN MANIFOLD

RAFIK NASRI AND MUSTAPHA DJAA

Laboratory of Geometry, Analysis, Control and Applications
Université de Saïda
BP138, En-Nasr, 20000 Saïda, Algeria

(Communicated by the associate editor name)

ABSTRACT. In this paper, we construct the symmetric tensor field $G_{f_1 f_2}$ and $h_{f_1 f_2}$ on a product manifold and we give conditions under which $G_{f_1 f_2}$ becomes a metric tensor, these tensors fields will be called the generalized warped product, and then we develop an expression of curvature for the connection of the generalized warped product in relation to those corresponding analogues of its base and fiber and warping functions. By constructing a frame field in $M_1 \times_{f_1 f_2} M_2$ with respect to the Riemannian metric $G_{f_1 f_2}$ and $h_{f_1 f_2}$, then we calculate the Laplacian–Beltrami operator of a function on a generalized warped product which may be expressed in terms of the local restrictions of the functions to the base and fiber. Finally, we conclude some interesting relationships between the geometry of the couples (M_1, g_1) and (M_2, g_2) and that of $(M_1 \times M_2, h_{f_1 f_2})$.

1. Introduction. The warped product provides a way to construct new pseudo-Riemannian manifolds from the given ones, see [6], [3] and [2]. This construction has useful applications in general relativity, in the study of cosmological models and black holes. It generalizes the direct product in the class of pseudo-Riemannian manifolds and it is defined as follows. Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds and let $f_1 : M_1 \rightarrow \mathbb{R}^*$ be a positive smooth function on M_1 , the warped product of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric tensor $g_{f_1} := \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$, where π_1 and π_2 are the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. The manifold M_1 is called the base of $(M_1 \times M_2, g_{f_1})$ and M_2 is called the fiber. The function f_1 is called the warping function.

The doubly warped product is construction in the class of pseudo-Riemannian manifolds generalizing the warped product and the direct product, it is obtained by homothetically distorting the geometry of each base $M_1 \times \{q\}$ and each fiber $\{p\} \times M_2$ to get a new "doubly warped" metric tensor on the product manifold and defined as follows. For $i \in \{1, 2\}$, let M_i be a pseudo-Riemannian manifold equipped with metric g_i , and $f_i : M_i \rightarrow \mathbb{R}^*$ be a positive smooth function on M_i . The well-known notion of doubly warped product manifold $M_1 \times_{f_1 f_2} M_2$ is defined as the product

2010 *Mathematics Subject Classification.* 53C21, 53C50.

Key words and phrases. warped products; generalized warped products; Ricci curvature; scalar curvature Laplacian–Beltrami operator.

manifold $M = M_1 \times M_2$ equipped with pseudo-Riemannian metric which is denoted by $g_{f_1 f_2}$, given by

$$g_{f_1 f_2} = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2.$$

When the warping functions $f_1 = 1$ or $f_2 = 1$ we obtain a warped product or direct product.

The paper is organized as follows. In section 2, we collect the basic material about Levi-Civita connection, horizontal and vertical lifts. In section 3, we consider the metric tensors g_1 and g_2 on manifolds M_1 and M_2 respectively and, for a smooth function f_i on M_i , $i = 1, 2$, we define the symmetric tensors fields $G_{f_1 f_2}$ and $h_{f_1 f_2}$ on $M_1 \times M_2$ relative to g_1 , g_2 and the warping functions f_1 , f_2 , then we give the condition under which $G_{f_1 f_2}$ becomes a metric tensor, this tensor field will be referred to as the generalized warped product metric, next, we define also its cometric and we compute the gradients of the lifts of f_1 , f_2 . Moreover, by constructing a frame field in $M_1 \times_{f_1 f_2} M_2$ with respect to the Riemannian metric $G_{f_1 f_2}$, then we calculate the Laplacian–Beltrami operator of a function on a generalized warped product which may be expressed in terms of the local restrictions of the functions to the base and fiber. To end this section, we conclude with some important relationships related to the harmonicity of function. In the final section, we compute the curvatures of generalized warped product $h_{f_1 f_2}$ and we conclude with some important relationships between the geometry of the triples (M_1, g_1) , (M_2, g_2) and that of $(M_1 \times M_2, h_{f_1 f_2})$.

2. Preliminaries.

2.1. Horizontal and vertical lifts. Throughout this paper M_1 and M_2 will be respectively m_1 and m_2 dimensional manifolds, $M_1 \times M_2$ the product manifold with the natural product coordinate system and $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ the usual projection maps.

We recall briefly how the calculus on the product manifold $M_1 \times M_2$ derives from that of M_1 and M_2 separately. For details see [6].

Let φ_1 in $C^\infty(M_1)$. The horizontal lift of φ_1 to $M_1 \times M_2$ is $\varphi_1^h = \varphi_1 \circ \pi_1$. One can define the horizontal lifts of tangent vectors as follows. Let $p_1 \in M_1$ and let $X_{p_1} \in T_{p_1} M_1$. For any $p_2 \in M_2$ the horizontal lift of X_{p_1} to (p_1, p_2) is the unique tangent vector $X_{(p_1, p_2)}^h$ in $T_{(p_1, p_2)}(M_1 \times M_2)$ such that $d_{(p_1, p_2)} \pi_1(X_{(p_1, p_2)}^h) = X_{p_1}$ and $d_{(p_1, p_2)} \pi_2(X_{(p_1, p_2)}^h) = 0$.

We can also define the horizontal lifts of vector fields as follows. Let $X_1 \in \Gamma(TM_1)$. The horizontal lift of X_1 to $M_1 \times M_2$ is the vector field $X_1^h \in \Gamma(T(M_1 \times M_2))$ whose value at each (p_1, p_2) is the horizontal lift of the tangent vector $(X_1)_{p_1}$ to (p_1, p_2) . For $(p_1, p_2) \in M_1 \times M_2$, we will denote the set of the horizontal lifts to (p_1, p_2) of all the tangent vectors of M_1 at p_1 by $L(p_1, p_2)(M_1)$. We will denote the set of the horizontal lifts of all vector fields on M_1 by $\mathfrak{L}(M_1)$.

The vertical lift φ_2^v of a function $\varphi_2 \in C^\infty(M_2)$ to $M_1 \times M_2$ and the vertical lift X_2^v of a vector field $X_2 \in \Gamma(TM_2)$ to $M_1 \times M_2$ are defined in the same way using the projection π_2 . Note that the spaces $\mathfrak{L}(M_1)$ of the horizontal lifts and $\mathfrak{L}(M_2)$ of the vertical lifts are vector subspaces of $\Gamma(T(M_1 \times M_2))$ but neither is invariant under multiplication by arbitrary functions $\varphi \in C^\infty(M_1 \times M_2)$.

Observe that if $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{m_1}}\}$ is the local basis of the vector fields (resp. $\{dx_1, \dots, dx_{m_1}\}$ is the local basis of 1-forms) relative to a chart (U, Φ) of M_1 and $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{m_2}}\}$ is the local basis of the vector fields (resp. $\{dy_1, \dots, dy_{m_2}\}$ the local basis of the 1-forms) relative to a chart (V, Ψ) of M_2 , then $\{(\frac{\partial}{\partial x_1})^h, \dots, (\frac{\partial}{\partial x_{m_1}})^h, (\frac{\partial}{\partial y_1})^v, \dots, (\frac{\partial}{\partial y_{m_2}})^v\}$ is the local basis of the vector fields (resp. $\{(dx_1)^h, \dots, (dx_{m_1})^h, (dy_1)^v, \dots, (dy_{m_2})^v\}$ is the local basis of the 1-forms) relative to the chart $(U \times V, \Phi \times \Psi)$ of $M_1 \times M_2$.

The following lemma will be useful later for our computations.

Lemma 2.1.

1. Let $\varphi_i \in C^\infty(M_i)$, $X_i, Y_i \in \Gamma(TM_i)$ and $\alpha_i \in \Gamma(T^*M_i)$, $i = 1, 2$. Let $\varphi = \varphi_1^h + \varphi_2^v$, $X = X_1^h + X_2^v$ and $\alpha, \beta \in \Gamma(T^*(M_1 \times M_2))$. Then
i/ For all $(i, I) \in \{(1, h), (2, v)\}$, we have

$$X_i^I(\varphi) = X_i(\varphi_i)^I, \quad [X, Y_i^I] = [X_i, Y_i]^I \quad \text{and} \quad \alpha_i^I(X) = \alpha_i(X_i)^I.$$

ii/ If for all $(i, I) \in \{(1, h), (2, v)\}$ we have $\alpha(X_i^I) = \beta(X_i^I)$, then $\alpha = \beta$.
2. Let ω_i and η_i be r -forms on M_i , $i = 1, 2$. Let $\omega = \omega_1^h + \omega_2^v$ and $\eta = \eta_1^h + \eta_2^v$. We have

$$d\omega = (d\omega_1)^h + (d\omega_2)^v \quad \text{and} \quad \omega \wedge \eta = (\omega_1 \wedge \eta_1)^h + (\omega_2 \wedge \eta_2)^v.$$

Proof. See [4]. □

Remark 1. Let X be a vector field on $M_1 \times M_2$, such that $d\pi_1(X) = \varphi(X_1 \circ \pi_1)$ and $d\pi_2(X) = \phi(X_2 \circ \pi_2)$, then $X = \varphi X_1^h + \phi X_2^v$.

3. About generalized warped products.

3.1. The generalized warped product. let $\psi : M \rightarrow N$ be a smooth map between smooth manifolds and let g be a metric on k -vector bundle (F, P_F) over N . The metric $g^\psi : \Gamma(\psi^{-1}F) \times \Gamma(\psi^{-1}F) \rightarrow C^\infty(M)$ on the pull-back $(\psi^{-1}F, P_{\psi^{-1}F})$ over M is defined by

$$g^\psi(U, V)(p) = g_{\psi(p)}(U_p, V_p), \quad \forall U, V \in \Gamma(\psi^{-1}F), \quad p \in M.$$

Given a linear connection ∇^N on k -vector bundle (F, P_F) over N , the pull-back connection $\overset{\psi}{\nabla}$ is the unique linear connection on the pull-back $(\psi^{-1}F, P_{\psi^{-1}F})$ over M such that

$$\overset{\psi}{\nabla}_X(W \circ \psi) = \nabla_{d\psi(X)}^N W, \quad \forall W \in \Gamma(F), \quad \forall X \in \Gamma(TM). \quad (1)$$

Further, let $U \in \psi^{-1}F$ and let $p \in M$, $X \in \Gamma(TM)$. Then

$$(\overset{\psi}{\nabla}_X U)(p) = (\nabla_{d\psi(X_p)}^N \tilde{U})(\psi(p)), \quad (2)$$

where $\tilde{U} \in \Gamma(F)$ with $\tilde{U} \circ \psi = U$.

Now, let π_i , $i=1,2$, be the usual projection of $M_1 \times M_2$ onto M_i , given a linear connection ∇ on vector bundle TM_i , the pull-back connection $\overset{\pi_i}{\nabla}$ is the unique linear connection on the pull-back $M_1 \times M_2 \rightarrow \pi_i^{-1}(TM_i)$ such that for each $Y_i \in \Gamma(TM_i)$, $X \in \Gamma(TM_1 \times M_2)$

$$\overset{\pi_i}{\nabla}_X(Y_i \circ \pi_i) = \overset{i}{\nabla}_{d\pi_i(X)} Y_i. \quad (3)$$

Further, let $(p_1, p_2) \in M_1 \times M_2$, $U \in \pi_i^{-1}(TM)$ and $X \in \Gamma(TM_1 \times M_2)$. Then

$$(\tilde{\nabla}_X U)(p_1, p_2) = (\tilde{\nabla}_{d\pi_i(X_{(p_1, p_2)})}^i \tilde{U})(p_i), \quad (4)$$

Now, we construct a symmetric tensor fields on product manifold and give the condition under which it becomes a tensor metric.

Let c be an arbitrary real number and let g_i , $(i = 1, 2)$ be a Riemannian metric tensors on M_i . Given a smooth positive function f_i on M_i , we define a symmetric tensor field on $M_1 \times M_2$ by

$$G_{f_1, f_2} = (f_2^v)^2 \pi_1^* g_1 + (f_1^h)^2 \pi_2^* g_2 + c f_1^h f_2^v df_1^h \odot df_2^v. \quad (5)$$

Where π_i , $(i = 1, 2)$ is the projection of $M_1 \times M_2$ onto M_i and

$$df_1^h \odot df_2^v = df_1^h \otimes df_2^v + df_2^v \otimes df_1^h.$$

For all $X, Y \in \Gamma(TM_1 \times M_2)$

$$\begin{aligned} G_{f_1, f_2}(X, Y) &= (f_2^v)^2 g_1^{\pi_1}(d\pi_1(X), d\pi_1(Y)) + (f_1^h)^2 g_2^{\pi_2}(d\pi_2(X), d\pi_2(Y)) \\ &\quad + c f_1^h f_2^v (X(f_1^h)Y(f_2^v) + X(f_2^v)Y(f_1^h)). \end{aligned}$$

It is the unique tensor fields such that for any $X_i, Y_i \in \Gamma(TM_i)$, $(i = 1, 2)$

$$G_{f_1 f_2}(X_i^I, Y_k^K) = \begin{cases} (f_{3-i}^J)^2 g_i(X_i, Y_i)^I, & \text{if } (i, I) = (k, K) \\ c f_i^I f_k^K X_i(f_i)^I Y_k(f_k)^K, & \text{otherwise} \end{cases} \quad (6)$$

where $(i, I), (k, K), (3-i, J) \in \{(1, h), (2, v)\}$.

We call G_{f_1, f_2} the generalized warped product relative to g_1, g_2 and the warping functions f_1, f_2 .

If either $f_1 \equiv 1$ or $f_2 \equiv 1$ but not both, then we obtain a singly warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a product manifold. If neither f_1 nor f_2 is constant and $c = 0$, then we have a nontrivial doubly warped product. If neither f_1 nor f_2 is constant and $c \neq 0$, then we have a nontrivial generalized warped product.

Now, Let us assume that (M_i, g_i) , $(i = 1, 2)$ is a smooth connected Riemannian manifold. The following proposition provides a necessary and sufficient condition for a symmetric tensor field G_{f_1, f_2} of type $(0, 2)$ of two Riemannian metrics to be a Riemannian metric.

Proposition 1. *Let (M_i, g_i) , $(i = 1, 2)$ be a Riemannian manifold and let f_i be a positive smooth function on M_i and c be an arbitrary real number. Then the symmetric tensor field $G_{f_1 f_2}$ is Riemannian metric on $M_1 \times M_2$ if and only if*

$$0 \leq c^2 g_1(\text{grad} f_1, \text{grad} f_1)^h g_2(\text{grad} f_2, \text{grad} f_2)^v < 1. \quad (7)$$

Proof. Let $\{e_1, \dots, e_{m_1}\}$ and $\{e_{m_1+1}, \dots, e_{m_1+m_2}\}$ be a local, orthonormal basis of the vector fields with respect to g_1 and g_2 on an open $O_1 \subset M_1$ and $O_2 \subset M_2$ respectively. The matrix of $G_{f_1 f_2}$ relative to

$$\{v_1 = \frac{1}{f_2^v} e_1^h, \dots, v_{m_1} = \frac{1}{f_2^v} e_{m_1}^h, v_{m_1+1} = \frac{1}{f_1^h} e_{m_1+1}^v, \dots, v_{m_1+m_2} = \frac{1}{f_1^h} e_{m_1+m_2}^v\}$$

has the form

$$\begin{pmatrix} D_1 & c f_1^h f_2^v E \\ c f_1^h f_2^v {}^t E & D_2 \end{pmatrix}. \quad (8)$$

Where $D_1 = (f_2^v)^2 I_{m_1}$, $D_2 = (f_1^h)^2 I_{m_2}$ and

$$E = \begin{pmatrix} e_1(f_1)^h e_{m_1+1}(f_2)^v & \cdots & e_1(f_1)^h e_{m_1+m_2}(f_2)^v \\ \vdots & \ddots & \vdots \\ e_{m_1}(f_1)^h e_{m_1+1}(f_2)^v & \cdots & e_{m_1}(f_1)^h e_{m_1+m_2}(f_2)^v \end{pmatrix}$$

We can write the matrix (8) as

$$\begin{pmatrix} I_{m_1} & O \\ c f_1^h f_2^v {}^t E D_1^{-1} & -(c f_1^h f_2^v)^2 {}^t E D_1^{-1} E + D_2 \end{pmatrix} \begin{pmatrix} D_1 & c f_1^h f_2^v E \\ O & I_{m_2} \end{pmatrix}. \quad (9)$$

So

$$\det \begin{pmatrix} D_1 & c f_1^h f_2^v E \\ c f_1^h f_2^v {}^t E & D_2 \end{pmatrix} = (f_1^h)^{2m_2} (f_2^v)^{2m_1} \det (I - c^2 {}^t E E).$$

and we compute

$$I - c^2 {}^t E E = - \begin{pmatrix} \lambda d_1^2 - 1 & \lambda d_1 d_2 & \lambda d_1 d_3 & \cdots & \lambda d_1 d_{m_2} \\ \lambda d_1 d_2 & \lambda d_2^2 - 1 & \lambda d_2 d_3 & \cdots & \lambda d_2 d_{m_2} \\ \vdots & \cdots & \ddots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \ddots & \cdots \\ \lambda d_1 d_{m_2} & \lambda d_2 d_{m_2} & \lambda d_3 d_{m_2} & \cdots & \lambda d_{m_2}^2 - 1 \end{pmatrix}.$$

Where $\lambda = c^2 \sum_{i=1}^{m_1} (e_i(f_1)^h)^2$ and $d_j = e_{m_1+j}(f_2)^v$.

By a straightforward long computation using a limited recurrence gives

$$(P_m) \left\{ \begin{array}{l} \det \begin{pmatrix} d_{11}-1 & d_{12} & d_{13} & \cdots & d_{1m} \\ d_{21} & d_{22}-1 & d_{23} & \cdots & d_{2m} \\ \vdots & \cdots & \ddots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \ddots & \cdots \\ d_{m1} & d_{m2} & d_{m3} & \cdots & d_{mm}-1 \end{pmatrix} = (-1)^m \left(1 - \lambda \sum_{j=1}^m d_j^2 \right), \\ \\ \det \begin{pmatrix} d_{i1} & d_{i2} & d_{i3} & \cdots & \cdots & d_{ii-1} & d_{ii+1} & \cdots & \cdots & d_{im} \\ d_{21} & d_{22}-1 & d_{23} & \cdots & \cdots & d_{2i-1} & d_{2i+1} & \cdots & \cdots & d_{2m} \\ \vdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{i-11} & d_{i-12} & d_{i-13} & \cdots & d_{i-1i-2} & \lambda d_{i-1i-1} - 1 & d_{i-1i+1} & \cdots & \cdots & d_{i-1m} \\ d_{i+11} & d_{i+12} & d_{i+13} & \cdots & d_{i+1i-2} & d_{i+1i-1} & d_{i+1i+1} - 1 & d_{i+1i+2} & \cdots & d_{i+1m} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ d_{m1} & d_{m2} & d_{m3} & \cdots & \cdots & d_{mi-1} & d_{mi+1} & \cdots & \cdots & \lambda d_{mm} - 1 \end{pmatrix} = (-1)^m \lambda d_1 d_i. \end{array} \right.$$

Where $d_{ij} = \lambda d_i d_j$.

So,

$$\begin{aligned} \det(\mathcal{M}_{f_1 f_2}) &= \det \begin{pmatrix} D_1 & c f_1^h f_2^v E \\ c f_1^h f_2^v {}^t E & D_2 \end{pmatrix} \\ &= (f_1^h)^{2m_2} (f_2^v)^{2m_1} \{ 1 - c^2 g_1(\text{grad} f_1, \text{grad} f_1)^h g_2(\text{grad} f_2, \text{grad} f_2)^v \}, \end{aligned} \quad (10)$$

where m_i ($i = 1, 2$) is the dimension of M_i . Since, f_1 and f_2 are non-constant smooth functions, then the proposition follows. \square

Corollary 1. *If the symmetric tensor field G_{f_1, f_2} of type $(0, 2)$ on $M_1 \times M_2$ is degenerate, then for any $i \in \{1, 2\}$, $g_i(\text{grad}f_i, \text{grad}f_i)$ is positive constant k_i with*

$$k_i = \frac{1}{c^2 k_{(3-i)}}.$$

Proof. Note that if G_{f_1, f_2} is degenerate then c is non-zero real number, f_1, f_2 is nonconstant smooth functions on M_1, M_2 respectively and we have

$$c^2 g_1(\text{grad}f_1, \text{grad}f_1)^h g_2(\text{grad}f_2, \text{grad}f_2)^v = 1.$$

Since $g_i(\text{grad}f_i, \text{grad}f_i)$ depend only on M_i , ($i = 1, 2$) we conclude that $g_i(\text{grad}f_i, \text{grad}f_i)$ is constant. \square

Remark 2. Under the same assumptions as in Proposition 1, if f_1, f_2 are non-constant smooth functions on M_1, M_2 respectively and φ is smooth function on $M_1 \times M_2$ that satisfies $\frac{-1}{\|\text{grad}f_1\|^h \|\text{grad}f_2\|^v} < \varphi < \frac{1}{\|\text{grad}f_1\|^h \|\text{grad}f_2\|^v}$, then the symmetric tensor fields

$$G_{f_1, f_2} = (f_2^v)^2 \pi_1^* g_1 + (f_1^h)^2 \pi_2^* g_2 + \varphi f_1^h f_2^v df_1^h \odot df_2^v.$$

is Riemannian metric on $M_1 \times M_2$.

In all what follows, we suppose that f_1 and f_2 satisfies the inequality (7).

Lemma 3.1. *Let X be an arbitrary vector field of $M_1 \times M_2$, if there exist $\varphi_i, \psi_i \in C^\infty(M_i)$ and $X_i, Y_i \in \Gamma(TM_i)$, ($i = 1, 2$) such that*

$$\begin{cases} G_{f_1 f_2}(X, Z_1^h) = G_{f_1 f_2}(\varphi_2^v X_1^h + \varphi_1^h X_2^v, Z_1^h), \\ G_{f_1 f_2}(X, Z_2^v) = G_{f_1 f_2}(\psi_2^v Y_1^h + \psi_1^h Y_2^v, Z_2^v). \end{cases} \quad \forall Z_i \in \Gamma(TM_i),$$

Then we have,

$$\begin{aligned} X &= \varphi_2^v X_1^h + \psi_1^h Y_2^v + c f_1^h f_2^v \{ \psi_2^v Y_1(f_1)^h - \varphi_2^v X_1(f_1)^h \} \text{grad}(f_2^v) \\ &\quad - c f_1^h f_2^v \{ \psi_1^h Y_2(f_2)^v - \varphi_1^h X_2(f_2)^v \} \text{grad}(f_1^h) \end{aligned} \quad (11)$$

Proof. At first, we put

$$B = X - \varphi_2^v X_1^h - \psi_1^h Y_2^v \quad \text{and} \quad Z = Z_1^h + Z_2^v.$$

It suffices to observe that

$$\begin{aligned} \frac{-1}{c f_1^h f_2^v} G_{f_1 f_2}(B, Z) &= \frac{1}{c f_1^h f_2^v} \{ G_{f_1 f_2}(\psi_1^h Y_2^v - \varphi_1^h X_2^v, Z_1^h) + G_{f_1 f_2}(\varphi_2^v X_1^h - \psi_2^v Y_1^h, Z_2^v) \} \\ &= \{ (\psi_1^h Y_2^v(f_2^v) - \varphi_1^h X_2^v(f_2^v)) Z_1^h(f_1^h) + (\varphi_2^v X_1^h(f_1^h) - \psi_2^v Y_1^h(f_1^h)) Z_2^v(f_2^v) \} \\ &= \sum_{i=1}^2 (-1)^i G_{f_1 f_2}(\{ \psi_{3-i}^J Y_i(f_i)^I - \varphi_{3-i}^J X_i(f_i)^I \} \text{grad}(f_{3-i}^J), Z). \end{aligned}$$

With $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$. The result follows. \square

3.2. The Levi-Civita Connection.

Lemma 3.2. *Let (M_i, g_i) , $(i = 1, 2)$ be a Riemannian manifold. The gradient of the lifts f_1^h of f_1 and f_2^v of f_2 to $M_1 \times_{f_1, f_2} M_2$ w.r.t. G_{f_1, f_2} are*

$$\text{grad}(f_1^h) = \frac{1}{1 - c^2 b_1^h b_2^v} \left\{ \frac{1}{(f_2^v)^2} (\text{grad} f_1)^h - \frac{c b_1^h}{f_1^h f_2^v} (\text{grad} f_2)^v \right\}, \quad (12)$$

$$\text{grad}(f_2^v) = \frac{1}{1 - c^2 b_1^h b_2^v} \left\{ \frac{1}{(f_1^h)^2} (\text{grad} f_2)^v - \frac{c b_2^v}{f_1^h f_2^v} (\text{grad} f_1)^h \right\}, \quad (13)$$

where $b_i = \|\text{grad} f_i\|^2$ ($i=1, 2$).

Proof. Let $Z_i \in \Gamma(TM_i)$, $i = 1, 2$, then for $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$, we have,

$$G_{f_1 f_2}(\text{grad}(f_i^I), Z_i^I) = \frac{1}{(f_{3-i}^J)^2} G_{f_1 f_2}((\text{grad} f_i)^I, Z_i^I),$$

and

$$G_{f_1 f_2}(\text{grad}(f_i^I), Z_{3-i}^J) = 0.$$

Therefor, the result follows by Equation (6) and Lemma 3.1. \square

Lemma 3.3. *Let (M_i, g_i) , $(i = 1, 2)$ be a Riemannian manifold and let φ_i be a smooth function on M_i . The gradient of the lifts φ_1^h of φ_1 and φ_2^v of φ_2 to $M_1 \times_{f_1, f_2} M_2$ w.r.t. G_{f_1, f_2} are*

$$\text{grad}(\varphi_1^h) = \left(\frac{1}{f_2^v} \right)^2 (\text{grad} \varphi_1)^h - c \frac{(f_1 \text{grad} \varphi_1(f_1))^h}{f_2^v} \text{grad}(f_2^v), \quad (14)$$

$$\text{grad}(\varphi_2^v) = \left(\frac{1}{f_1^h} \right)^2 (\text{grad} \varphi_2)^v - c \frac{(f_2 \text{grad} \varphi_2(f_2))^v}{f_1^h} \text{grad}(f_1^h), \quad (15)$$

Proof. Let $Z_i \in \Gamma(TM_i)$, $(i = 1, 2)$ then for $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$, we have,

$$G_{f_1 f_2}(\text{grad}(\varphi_i^I), Z_i^I) = \frac{1}{(f_{3-i}^J)^2} G_{f_1 f_2}((\text{grad} \varphi_i)^I, Z_i^I),$$

and

$$G_{f_1 f_2}(\text{grad}(\varphi_i^I), Z_{3-i}^J) = 0.$$

Therefor, the result follows by Equation (6) and Lemma 3.1. \square

Proposition 2. *Let (M_1, g_1) , $(i = 1, 2)$ be a pseudo-Riemannian manifold and let $f_i : M_i \rightarrow \mathbb{R}_+^*$, be a positive smooth function. The cometric $\tilde{G}_{f_1 f_2}$ of $G_{f_1 f_2}$ is given by*

$$\begin{aligned} \tilde{G}_{f_1 f_2} &= \left(\frac{1}{f_2^v} \right)^2 \tilde{g}_1^h + \left(\frac{1}{f_1^h} \right)^2 \tilde{g}_2^v + \frac{1}{1 - c^2 b_1^h b_2^v} \left\{ \frac{c^2 b_2^v}{(f_2^v)^2} (\text{grad} f_1)^h \odot (\text{grad} f_1)^h \right. \\ &\quad \left. + \frac{c^2 b_1^h}{(f_1^h)^2} (\text{grad} f_2)^v \odot (\text{grad} f_2)^v - \frac{c}{f_1^h f_2^v} (\text{grad} f_1)^h \odot (\text{grad} f_2)^v \right\}. \end{aligned} \quad (16)$$

It is the unique tensor fields such that

$$\tilde{G}_{f_1 f_2}(\alpha_i^I, \beta_k^K) = \begin{cases} \frac{1}{(f_j^J)^2} \left\{ \tilde{g}_i(\alpha_i, \beta_i)^I + \frac{c^2 b_j^J}{1 - c^2 b_1^h b_2^v} \tilde{g}_i(\alpha_i, df_i)^I \tilde{g}_i(\beta_i, df_i)^I \right\}, & \text{if } i = k \\ \frac{-c}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} \tilde{g}_i(\alpha_i, df_i)^I \tilde{g}_k(\beta_k, df_k)^K. & \text{if } i \neq k \end{cases} \quad (17)$$

for any $\alpha_i, \beta_i \in \Gamma(T^*M_i)$ ($i = 1, 2$ and $j = 3 - i$). Where \tilde{g}_i ($i = 1, 2$) is the cometrics of g_i and $(i, I), (k, K), (j, J) \in \{(1, h), (2, v)\}$.

Proof. A direct computation using Equation 6, the definition of the musical isomorphisms and

$$\sharp_{f_1 f_2}(\alpha_i^I) = \left(\frac{1}{f_{3-i}^J} \right)^2 \left(\sharp_{g_i}(\alpha_i) \right)^I - \frac{c f_i^I}{f_{3-i}^J} \tilde{g}_i(\alpha_i, df_i)^h \text{grad}(f_{3-i}^J),$$

for $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$, leads to gives (17). \square

let us compute the Levi-Civita connection of $M_1 \times_{f_1 f_2} M_2$ associated with the metric $G_{f_1 f_2}$ in terms of the Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ associated with the metrics g_1 and g_2 respectively.

Proposition 3. *Let (M_i, g_i) , $(i = 1, 2)$ be a Riemannian manifold. Then we have*

$$\nabla_{X_1^h} Y_1^h = (\overset{1}{\nabla}_{X_1} Y_1)^h + f_2^v B_{f_1}(X_1, Y_1)^h \text{grad}(f_2^v) \quad (18)$$

$$\nabla_{X_2^v} Y_2^v = (\overset{2}{\nabla}_{X_2} Y_2)^v + f_1^h B_{f_2}(X_2, Y_2)^h \text{grad}(f_1^h) \quad (19)$$

$$\begin{aligned} \nabla_{X_1^h} Y_2^v = \nabla_{Y_2^v} X_1^h &= -c X_1(f_1)^h Y_2(f_2)^v \{ f_2^v \text{grad}(f_1^h) + f_1^h \text{grad}(f_2^v) \} \\ &\quad + (Y_2(\ln f_2))^v X_1^h + (X_1(\ln f_1))^h Y_2^v, \end{aligned} \quad (20)$$

Where B_{f_i} , $(i = 1, 2)$ the symmetric $(0, 2)$ tensor field of f_i given by

$$B_{f_i}(X_i, Y_i) = c f_i^H H^{f_i}(X_i, Y_i) + c X_i(f_i) Y_i(f_i) - g_i(X_i, Y_i),$$

H^{f_i} is the Hessian of f_i .

Proof. Let $X_i, Y_i, Z_i \in \Gamma(TM_i)$, $i = 1, 2$. For any $(i, I), (k, K) \in \{(1, h), (2, v)\}$ we have

$$\begin{aligned} 2G_f(\nabla_{X_i^I} Y_i^I, Z_k^K) &= X_i^I(G_{f_1 f_2}(Y_i^I, Z_k^K)) + Y_i^I(G_{f_1 f_2}(X_i^I, Z_k^K)) - Z_k^K(G_{f_1 f_2}(X_i^I, Y_i^I)) \\ &\quad + G_{f_1 f_2}([X_i^I, Y_i^I], Z_k^K) + G_{f_1 f_2}([Z_k^K, X_i^I], Y_i^I) + G_{f_1 f_2}([Z_k^K, Y_i^I], X_i^I). \end{aligned} \quad (21)$$

1. Taking $(i, I) = (k, K)$ in this formula, using Formula (6) and Lemma 2.1, we get

$$2G_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_i^I) = 2(f_{3-i}^J)^2 (g_i(\overset{i}{\nabla}_{X_i} Y_i, Z_i))^I,$$

and using (6) again, we get

$$G_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_i^I) = G_{f_1 f_2}((\overset{i}{\nabla}_{X_i} Y_i)^I, Z_i^I).$$

Similarly, taking $(i, I) \neq (k, K)$, we get

$$G_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_k^K) = \left(\frac{c X_i(f_i) Y_i(f_i) - g_i(X_i, Y_i)}{(f_i^2)} \right)^I G_{f_1 f_2}((f_k \text{grad} f_k)^K, Z_k^K)$$

The result then follows by Lemma 3.1.

2. Taking $i \neq k$.

At first, since ∇ is torsion-free we have $\nabla_{Y_k^K} X_i^I = \nabla_{X_i^I} Y_k^K + [X_i^I, Y_k^K]$. By Lemma 2.1, we have $[X_i^I, Y_k^K] = 0$. This implies that $\nabla_{X_i^I} Y_k^K = \nabla_{Y_k^K} X_i^I$.

Using Formula (6) and Lemma 2.1, we get

$$G_{f_1 f_2}(\nabla_{X_i^I} Y_j^K, Z_i^I) = G_{f_1 f_2}((\frac{Y_k(f_k)}{f_k})^K X_i^I, Z_i^I),$$

and

$$G_{f_1 f_2}(\nabla_{X_i^I} Y_k^K, Z_k^K) = G_{f_1 f_2}((\frac{X_i(f_i)}{f_i})^I Y_k^K, Z_k^K).$$

Thus the result follows by Lemma 3.1. \square

3.3. The Laplacian of the lifts to M_1 and M_2 .

Theorem 3.4. *On a generalized warped product $(M_1 \times_{f_1 f_2} M_2, G_{f_1 f_2})$ with $m_1 = \dim M_1$ and $m_2 = \dim M_2$, Let $f_1 : M_1 \rightarrow \mathbb{R}$ and $f_2 : M_1 \rightarrow \mathbb{R}$ be a smooth functions. Then the Laplacian of the horizontal lift $f_1 \circ \pi_1$ of f_1 (resp. vertical lift $f_2 \circ \pi_2$ of f_2) to $M_1 \times_{f_1 f_2} M_2$ is given by*

$$\Delta(f_1^h) = \frac{1}{f_2^v(1 - c^2 b_1^h b_2^v)} \left\{ \frac{1}{f_2^v} (\Delta_1(f_1))^h - \frac{c b_1^h}{f_1^h} (\Delta_2(f_2))^v + \frac{b_1^h (c(1 - m_1) b_2^v + m_2)}{f_1^h f_2^v} \right\} \\ + \frac{c^2}{2 f_2^v (1 - c^2 b_1^h b_2^v)^2} \left\{ \frac{b_2^v}{f_2^v} (\text{grad} f_1(b_1))^h - \frac{c(b_1^2)^h}{f_1^h} (\text{grad} f_2(b_2))^v \right\}. \quad (22)$$

$$\Delta(f_2^v) = \frac{1}{f_1^h(1 - c^2 b_1^h b_2^v)} \left\{ \frac{1}{f_1^h} (\Delta_2(f_2))^v - \frac{c b_2^v}{f_2^v} (\Delta_1(f_1))^h + \frac{b_2^v (c(1 - m_2) b_1^h + m_1)}{f_1^h f_2^v} \right\} \\ + \frac{c^2}{2 f_1^h (1 - c^2 b_1^h b_2^v)^2} \left\{ \frac{b_1^h}{f_1^h} (\text{grad} f_2(b_2))^v - \frac{c(b_2^2)^v}{f_2^v} (\text{grad} f_1(b_1))^h \right\}. \quad (23)$$

Where $b_i = \|\text{grad} f_i\|^2$ ($i=1,2$).

Lemma 3.5. *On $(M_1 \times_{f_1 f_2} M_2, G_{f_1 f_2})$, if $\{e_1, \dots, e_{m_1}\}$ is the local frame field with respect to the metric g_1 and $\{e_{m_1+1}, \dots, e_{m_1+m_2}\}$ is the local frame field with respect to the metric g_2 , then $\{u_1, \dots, u_{m_1}, u_{m_1+1}, \dots, u_{m_1+m_2}\}$ is the local frame field with respect to the metric $G_{f_1 f_2}$, where*

$$u_j = \begin{cases} \frac{1}{f_2^v} e_j^h, & j \in \{1, \dots, m_1\}; \\ \frac{c a_j^v}{(1 - c^2 b_1^h A_j^v)} \left\{ -\frac{1}{f_2^v} (\text{grad} f_1)^h + \frac{c b_1^h}{f_1^h} T_j^v \right\} + \frac{1}{f_1^h} e_j^v, & j \in \{m_1 + 1, \dots, m_1 + m_2\}. \end{cases} \quad (24)$$

And for $j \in \{m_1 + 1, \dots, m_1 + m_2\}$,

$$u_j = \frac{1}{\|u_j'\|} u_j', \quad \|u_j'\|^2 = \frac{1 - c^2 b_1^h A_{j+1}^v}{1 - c^2 b_1^h A_j^v}, \quad A_j = \sum_{i=m_1+1}^{j-1} a_i^2, \quad T_j = \sum_{i=m_1+1}^{j-1} a_i e_i, \quad a_i = e_i(f_2).$$

Proof. We know that $G_{f_1 f_2}$ is Riemannian metric if and only if $0 < 1 - b_1^h b_2^v$. Then if we choose $\{e_1, \dots, e_{m_1}\}$ to be a local, orthonormal basis of the vector fields with respect to g_1 on an open $O_1 \subset M_1$ and $\{e_{m_1+1}, \dots, e_{m_1+m_2}\}$ to be a local orthonormal basis of the vector fields with respect to the metric g_2 on an open $O_2 \subset M_2$, then the family

$$\{v_1 = \frac{1}{f_2^v} e_1^h, \dots, v_{m_1} = \frac{1}{f_2^v} e_{m_1}^h, v_{m_1+1} = \frac{1}{f_1^h} e_{m_1+1}^v, \dots, v_{m_1+m_2} = \frac{1}{f_1^h} e_{m_1+m_2}^v\}$$

is a local basis of the vector fields with respect to $G_{f_1 f_2}$ on an open $O_1 \times O_2 \subset M_1 \times M_2$.

The gradient of f_1 (resp. f_2) and its norm $\|\text{grad} f_1\|$ (resp. $\|\text{grad} f_2\|$) can be written as

$$\text{grad} f_1 = \sum_{k=1}^{m_1} e_k(f_1) e_k, \quad \|\text{grad} f_1\|^2 = \sum_{k=1}^{m_1} (e_k(f_1))^2 \quad (25)$$

$$(rep. \ grad f_2 = \sum_{i=m_1+1}^{m_1+m_2} a_i e_i, \quad \|grad f_2\|^2 = \sum_{i=m_1+1}^{m_1+m_2} a_i^2). \quad (26)$$

$G_{f_1 f_2}$ is positive definite, which implies that

$$1 - c^2 b_2^h \sum_{k=1}^l (a_k^h)^2 > 0, \quad \forall l \in \{1, \dots, m_1\}, \quad (27)$$

and

$$1 - c^2 b_1^h \sum_{i=m_1+1}^j (a_i^v)^2 > 0, \quad \forall j \in \{m_1 + 1, \dots, m_1 + m_2\}. \quad (28)$$

For the proof of the lemma it is actually almost the most interesting result because it provides an algorithm for constructing $\{u_1, \dots, u_{m_1}, u_{m_1+1}, \dots, u_{m_1+m_2}\}$ from the family $\{e_1, \dots, e_{m_1}\}$ et $\{e_{m_1+1}, \dots, e_{m_1+m_2}\}$.

To do so, we use a limited recurrence (The Gram schmidt process).

At first, we put $u'_1 = v_1$ and $u_1 = \frac{v_1}{\|v_1\|}$. For $j \in \{2, \dots, m_1, m_1 + 1, \dots, m_1 + m_2\}$,

$$u'_j = v_j - \sum_{i=1}^{j-1} G_{f_1 f_2}(v_j, u_i) u_i \quad \text{and} \quad u_j = \frac{u'_j}{\|u'_j\|}. \quad (29)$$

By virtue of (29), a straightforward calculation using (25) and (26) gives

$$u_k = \frac{1}{f_2^v} e_k^h, \quad \|u_k\| = 1 \quad \forall k \in \{1, \dots, m_1\},$$

for all $j \in \{m_1, \dots, m_1 + m_2\}$, we have

$$\begin{aligned} u'_j &= \frac{-ca_j^v}{f_2^v \left(1 - c^2 b_1^h \sum_{i=m_1+1}^{j-1} (a_i^v)^2\right)} (grad f_1)^h + \frac{1}{f_1^h} e_j^v \\ &+ \frac{c^2 b_1^h a_j^v}{f_1^h \left(1 - c^2 b_1^h \sum_{i=m_1+1}^{j-1} (a_i^v)^2\right)} \left(\sum_{i=m_1+1}^{j-1} a_i e_i\right)^v, \end{aligned}$$

and

$$\|u'_j\| = \sqrt{\frac{\left(1 - c^2 b_1^h \sum_{i=m_1+1}^j (a_i^v)^2\right)}{\left(1 - c^2 b_1^h \sum_{i=m_1+1}^{j-1} (a_i^v)^2\right)}}.$$

□

Remark 3. With the notations above, we have

1) T_{m_1+1} is the zero vector field on M_2 , A_{m_1+1} is the zero function on M_2 and $A_{m_1+m_2}$ is the care of the gradient of f_2 .

2) For any $j \in \{m_1 + 1, \dots, m_1 + m_2\}$

$$\begin{cases} T_j(f_2) = A_j = g_2(T_j, T_j), \\ u'_j(f_1^h) = -\frac{cb_1^h}{f_2^v} \left(\frac{a_j^v}{1 - c^2 b_1^h A_j^v}\right) = -\frac{cf_1^h b_1^h}{f_2^v} u'_j(f_2^v), \\ u_j(f_1^h) = -\frac{cf_1^h b_1^h}{f_2^v} u_j(f_2^v), \end{cases} \quad (30)$$

Lemma 3.6. *With the notations above, we have*

$$\frac{1}{1 - c^2 b_1^h b_2^v} (grad f_2)^v = c^2 b_1^h \sum_{j=m_1+1}^{m_1+m_2} \left(\frac{a_j^v}{\|u'_j\| (1 - c^2 b_1^h A_j^v)} \right)^2 T_j^v + \sum_{j=m_1+1}^{m_1+m_2} \left(\frac{a_j^v}{\|u'_j\|^2 (1 - c^2 b_1^h A_j^v)} \right) e_j^v. \quad (31)$$

And

$$\frac{b_2^v}{1 - c^2 b_1^h b_2^v} = c^2 b_1^h \sum_{j=m_1+1}^{m_1+m_2} \left(\frac{a_j^v \sqrt{A_j^v}}{\|u'_j\| (1 - c^2 b_1^h A_j^v)} \right)^2 + \sum_{j=m_1+1}^{m_1+m_2} \left(\frac{(a_j^v)^2}{\|u'_j\|^2 (1 - c^2 b_1^h A_j^v)} \right). \quad (32)$$

Proof. From Lemma 3.5, $\{u_1, \dots, u_{m_1}, u_{m_1+1}, \dots, u_{m_1+m_2}\}$ is the local frame field with respect to the metric $G_{f_1 f_2}$, then

$$\begin{aligned} grad(f_1^h) &= \sum_{j=1}^{m_1+m_2} u_j(f_1^h) u_j = \left(\frac{1}{f_2^v} \right)^2 \sum_{j=1}^{m_1} e_j^h(f_1^h) e_j^h + \sum_{j=m_1+1}^{m_1+m_2} u_j(f_1^h) u_j \\ &= \left(\frac{1}{f_2^v} \right)^2 (grad f_1)^h + \frac{c^2 b_1^h}{(f_2^v)^2} \sum_{j=m_1+1}^{m_1+m_2} \left(\frac{a_j}{\|u'_j\| (1 - c^2 b_1^h A_j^v)} \right)^2 (grad f_1)^h \\ &\quad - \frac{c^3 (b_1^2)^h}{f_1^h f_2^v} \sum_{j=m_1+1}^{m_1+m_2} \left(\frac{a_j}{\|u'_j\| (1 - c^2 b_1^h A_j^v)} \right)^2 T_j^v - \frac{c b_1^h}{f_1^h f_2^v} \sum_{j=m_1+1}^{m_1+m_2} \frac{a_j}{\|u'_j\|^2 (1 - c^2 b_1^h A_j^v)} e_j^v. \end{aligned}$$

On the other hand, by (49), we have also,

$$\begin{aligned} \left(\frac{c b_1^h}{f_2^v} \right)^2 \sum_{j=m_1+1}^{m_1+m_2} \left(\frac{a_j}{\|u'_j\| (1 - c^2 b_1^h A_j^v)} \right)^2 &= \sum_{j=1}^{m_1+m_2} (u_j(f_1^h))^2 - \left(\frac{1}{f_2^v} \right) \left(\sum_{k=1}^{m_1} (e_k(f_1))^2 \right)^h \\ &= \|grad(f_1^h)\|^2 - \left(\frac{\|grad f_1\|^h}{f_2^v} \right)^2 \\ &= \frac{c^2 (b_1^2)^h b_2^v}{(f_2^v)^2 (1 - c^2 b_1^h b_2^v)}. \end{aligned}$$

Substituting in the previous equation then leads to the required result.

The second assertion can be calculated by applying Equation (31) the function f_2^v . \square

Lemma 3.7. *With the notations above, we have, for all $j \in \{m_1+1, \dots, m_1+m_2\}$*

$$\frac{1}{1 - c^2 b_1^h A_j^v} + c^2 b_1^h \sum_{i=j}^{m_1+m_2} \frac{(a_i^v)^2}{(1 - c^2 b_1^h A_i^v)(1 - b_1^h A_{i+1}^v)} = \frac{1}{1 - c^2 b_1^h b_2^v}, \quad (33)$$

$$\frac{(1 - c^2 b_1^h A_{j+1}^v)(1 - c^2 b_1^h A_{j-1}^v) + (c^2 b_1^h a_j^v a_{j-1}^v)^2}{(1 - c^2 b_1^h A_j^v)(1 - c^2 b_1^h (A_{j+1}^v - (a_{j-1}^v)^2))} = 1, \quad (34)$$

and

$$\frac{1}{\|u'_j\|^2} + (c^2 b_1^h a_j^v)^2 \sum_{i=j+1}^{m_1+m_2} \frac{(a_i^v)^2}{(1 - c^2 b_1^h A_i^v)(1 - c^2 b_1^h A_{i+1}^v)} = \frac{1 - c^2 b_1^h (b_2^v - (a_j^v)^2)}{1 - c^2 b_1^h b_2^v}. \quad (35)$$

Proof. Firstly, if we put $B_j = 1 - c^2 b_1^h A_j$ and $C^{i,i+1} = B_j \dots B_{i-1} B_{i+2} \dots B_{m_1+m_2+1}$ with $i \geq j$, then for any $j \in \{m_1+1, \dots, m_1+m_2\}$, the equation $\frac{1}{B_j} + c^2 b_1^h \sum_{i=j}^{m_1+m_2} \frac{(a_i^v)^2}{B_i B_{i+1}}$ becomes

$$\frac{\prod_{k=j+1}^{m_1+m_2+1} B_k + c^2 b_1^h \sum_{i=j}^{m_1+m_2} (a_i^v)^2 C^{i,i+1}}{\prod_{k=j}^{m_1+m_2+1} B_k},$$

furthermore,

$$\prod_{k=j+1}^{m_1+m_2+1} B_k + c^2 b_1^h \sum_{i=j}^{m_1+m_2} (a_i^v)^2 C^{i,i+1} = C^{j,j+1} (B_{j+1} + c^2 b_1^h (a_j^v)^2) + c^2 b_1^h \sum_{i=j+1}^{m_1+m_2} (a_i^v)^2 C^{i,i+1}.$$

Therefore, using the fact that $B_j C^{j,j+1} = B_{j+2} C^{j+1,j+2}$, $B_{j+1} + c^2 b_1^h (a_j^v)^2 = B_j$ and by induction, we have

$$\prod_{k=j+1}^{m_1+m_2+1} B_k + c^2 b_1^h \sum_{i=j}^{m_1+m_2} (a_i^v)^2 C^{i,i+1} = \prod_{k=j+1}^{m_1+m_2-1} B_k (B_{m_1+m_2+1} + c^2 b_1^h (a_{m_1+m_2}^v)^2).$$

Accordingly,

$$\frac{1}{B_j} + c^2 b_1^h \sum_{i=j}^{m_1+m_2} \frac{(a_i^v)^2}{B_{i+1} B_i} = \frac{\prod_{k=j+1}^{m_1+m_2-1} B_k (B_{m_1+m_2+1} + c^2 b_1^h (a_{m_1+m_2}^v)^2)}{\prod_{k=j}^{m_1+m_2+1} B_k} = \frac{1}{1 - c^2 b_1^h b_2^h}.$$

For $j \in \{m_1+2, \dots, m_1+m_2\}$, we have

$$\begin{aligned} B_{j+1} B_{j-1} + c^4 (b_1^2)^h (a_j^v a_{j-1}^v)^2 &= B_{j+1} (B_j + c^2 b_1^h (a_{j-1}^v)^2) + c^4 (b_1^2)^h (a_j^v a_{j-1}^v)^2 \\ &= B_{j+1} B_j + c^2 b_1^h (a_{j-1}^v)^2 (B_{j+1} + c^2 b_1^h (a_j^v)^2) \\ &= B_j (B_{j+1} + c^2 b_1^h (a_{j-1}^v)^2) \end{aligned}$$

The second assertion is true.

The third assertion follows from Equations (33) and (34). \square

Proof of Theorem 3.4

The proof of the theorem is a very long calculation. that will be omitted here.

Now, we calculate the Laplacian of the lifts f_1^h of f_1 , using Lemma 3.5.

$$\begin{aligned} \Delta(f_1^h) &= \sum_{j=1}^{m_1+m_2} G_{f_1 f_2}(\nabla_{u_j} \text{grad}(f_1^h), u_j) \\ &= \sum_{j=1}^{m_1} G_{f_1 f_2}(\nabla_{u_j} \text{grad}(f_1^h), u_j) + \sum_{j=m_1+1}^{m_1+m_2} G_{f_1 f_2}(\nabla_{u_j} \text{grad}(f_1^h), u_j) \\ &= \left(\frac{1}{f_2^v}\right)^2 \sum_{j=1}^{m_1} G_{f_1 f_2}(\nabla_{e_j^h} \text{grad}(f_1^h), e_j^h) + \sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u_j'} \text{grad}(f_1^h), u_j') \quad (36) \end{aligned}$$

Calculate, the first term on the right-hand side of the last equation above

$$\begin{aligned} G_{f_1 f_2}(\nabla_{e_j^h} \text{grad}(f_1^h), e_j^h) &= e_j^h(e_j^h(f_1^h)) - (\nabla_{e_j^h} e_j^h)(f_1^h) \\ &= \left(g_1(\nabla_{e_j} \text{grad} f_1, e_j) \right)^h - f_2^v(K_{f_1}(e_j, e_j))^h \text{grad}(f_2^v)(f_1^h) \\ &= \left(g_1(\nabla_{e_j} \text{grad} f_1, e_j) \right)^h \{1 - c f_1^h f_2^v \text{grad}(f_2^v)(f_1^h)\} \\ &\quad + f_2^v \text{grad}(f_2^v)(f_1^h) \{1 - c(e_j(f_1))^2\}^h. \end{aligned}$$

From this formula and a straightforward calculation, we obtain

$$\sum_{j=1}^{m_1} G_{f_1 f_2}(\nabla_{e_j^h} \text{grad}(f_1^h), e_j^h) = \frac{1}{(1 - c^2 b_1^h b_2^v)} \left\{ (\Delta_1(f_1))^h + \frac{c b_1^h b_2^v}{f_1} (c b_1^h - m_1) \right\}. \quad (37)$$

Calculate, the second term on the right-hand side of Equation (36). Straightforward calculation using Lemmas 3.6 and 3.7 gives

$$\begin{aligned} \sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u_j'} \text{grad}(f_1^h), u_j') &= \frac{c^2 b_2^v}{(f_2^v)^2 (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{grad} f_1)^h} \text{grad}(f_1^h), (\text{grad} f_1)^h) \\ &\quad - \frac{2c}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{grad} f_1)^h} \text{grad}(f_1^h), (\text{grad} f_2)^v) + \left(\frac{1}{f_1^h}\right)^2 \left[\sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u_j'\|^2} \{ \right. \\ &\quad \left. \left(\frac{c^2 b_1^h a_j^v}{1 - c^2 b_1^h A_j} \right)^2 G_{f_1 f_2}(\nabla_{T_j^v} \text{grad}(f_1^h), T_j^v) + \frac{2c^2 b_1^h a_j^v}{1 - c^2 b_1^h A_j} G_{f_1 f_2}(\nabla_{T_j^v} \text{grad}(f_1^h), e_j^v) \right. \\ &\quad \left. \left. + G_{f_1 f_2}(\nabla_{e_j^v} \text{grad}(f_1^h), e_j^v) \right\} \right]. \end{aligned}$$

So,

$$\begin{aligned} \sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u_j'} \text{grad}(f_1^h), u_j') &= \frac{c^2 b_2^v}{(f_2^v)^2 (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{grad} f_1)^h} \text{grad}(f_1^h), (\text{grad} f_1)^h) \\ &\quad - \frac{2c}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{grad} f_1)^h} \text{grad}(f_1^h), (\text{grad} f_2)^v) - \frac{1}{(f_1^h)^2 (1 - c^2 b_1^h b_2^v)} \left\{ (1 - c^2 b_1^h b_2^v) \sum_{m_1+1}^{m_1+m_2} \nabla_{e_j^v} e_j^v(f_1^h) \right. \\ &\quad \left. + c^2 b_1^h \sum_{j=m_1+1}^{m_1+m_2} (a_j^v)^2 \nabla_{e_j^v} e_j^v(f_1^h) + 2c^2 b_1^h \sum_{m_1+1 \leq i < j \leq m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v(f_1^h) \right\}. \end{aligned}$$

Since ∇ is torsion-free and $[e_j^v, e_j^v](f_1^h) = 0$, we deduce that

$$2 \sum_{m_1+1 \leq i < j \leq m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v(f_1^h) + \sum_{m_1+1}^{m_1+m_2} (a_j^v)^2 \nabla_{e_j^v} e_j^v(f_1^h) = \sum_{m_1+1 \leq i, j \leq m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v(f_1^h).$$

So

$$\begin{aligned} \sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2}(\nabla_{u_j'} \text{grad}(f_1^h), u_j') &= \frac{c^2 b_2^v}{(f_2^v)^2 (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{grad} f_1)^h} \text{grad}(f_1^h), (\text{grad} f_1)^h) \\ &\quad - \frac{2c}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2}(\nabla_{(\text{grad} f_1)^h} \text{grad}(f_1^h), (\text{grad} f_2)^v) - \left(\frac{1}{f_1^h}\right)^2 \left\{ \sum_{j=m_1+1}^{m_1+m_2} \nabla_{e_j^v} e_j^v(f_1^h) \right. \\ &\quad \left. + \frac{c^2 b_1^h}{1 - c^2 b_1^h b_2^v} \sum_{m_1+1 \leq i, j \leq m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v(f_1^h) \right\}. \end{aligned}$$

Since ∇ is compatible with $G_{f_1 f_2}$ and

$$\sum_{m_1+1 \leq i, j \leq m_1+m_2} a_i^v a_j^v \nabla_{e_i^v} e_j^v(f_1^h) = -G_{f_1 f_2}(\nabla \text{grad}(f_1^h), (\text{grad} f_2)^v)$$

then

$$\begin{aligned} \sum_{j=m_1+1}^{m_1+m_2} \frac{1}{\|u'\|^2} G_{f_1 f_2} \left(\nabla_{u'_j} \text{grad}(f_1^h), u'_j \right) &= \frac{c^2 b_2^v}{(f_2^v)^2 (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2} \left(\nabla \text{grad}(f_1^h), (\text{grad} f_1)^h \right) \\ &\quad - \frac{2c}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2} \left(\nabla \text{grad}(f_1^h), (\text{grad} f_2)^v \right) - \left(\frac{1}{f_1^h} \right)^2 \sum_{j=m_1+1}^{m_1+m_2} \nabla_{e_j^v} e_j^v(f_1^h) \\ &\quad + \frac{c^2 b_1^h}{(f_1^h)^2 (1 - c^2 b_1^h b_2^v)} G_{f_1 f_2} \left(\nabla \text{grad}(f_1^h), (\text{grad} f_2)^v \right). \end{aligned}$$

Using Proposition 3, we obtain

$$\begin{aligned} \Delta(f_1^h) &= \frac{1}{f_2^v (1 - c^2 b_1^h b_2^v)} \left\{ \frac{1}{f_2^v} (\Delta_1(f_1))^h - \frac{c b_1^h}{f_1^h} (\Delta_2(f_2))^v + \frac{b_1^h (c(1 - m_1) b_2^v + m_2)}{f_1^h f_2^v} \right\} \\ &\quad + \frac{c^2}{2 f_2^v (1 - c^2 b_1^h b_2^v)^2} \left\{ \frac{b_2^v}{f_2^v} (\text{grad} f_1(b_1))^h - \frac{c(b_1^h)^2}{f_1^h} (\text{grad} f_2(b_2))^v \right\}. \end{aligned}$$

For the Laplacian of f_2^v , just take $\{w_1, \dots, w_{m_2}, w_{m_2+1}, \dots, w_{m_2+m_1}\}$ the local frame field with respect to the metric $G_{f_1 f_2}$, where

$$W'_j = \begin{cases} \frac{1}{f_1^v} e_j^v, & j \in \{1, \dots, m_2\}; \\ \frac{1}{f_2^h} \left(\frac{c^2 b_2^v a_j^h}{(1 - c^2 b_2^v A_j^h)} T_j^h + e_j^h \right) - \frac{c a_j^h}{f_1^h (1 - c^2 b_2^v A_j^h)} (\text{grad} f_2)^v, & j \in \{m_2 + 1, \dots, m_2 + m_1\}. \end{cases} \quad (38)$$

And

$$w_j = \frac{1}{\|w'_j\|} w'_j, \quad \|w'_j\|^2 = \frac{1 - c^2 b_2^v A_{j+1}^h}{1 - c^2 b_2^v A_j^h}, \quad A_j = \sum_{i=m_2+1}^{j-1} a_i^2, \quad T_j = \sum_{i=m_2+1}^{j-1} a_i e'_i, \quad a_j = e_j(f_1).$$

such that $\{e'_1, \dots, e'_{m_2}\}$ is the local frame field with respect to the metric g_2 and $\{e'_{m_2+1}, \dots, e'_{m_2+m_1}\}$ is the local frame field with respect to the metric g_1 . Then

$$\begin{aligned} \Delta(f_2^v) &= \frac{1}{f_1^h (1 - c^2 b_1^h b_2^v)} \left\{ -\frac{c b_2^v}{f_2^v} (\Delta_1(f_1))^h + \frac{1}{f_1^h} (\Delta_2(f_2))^v + \frac{b_2^v (c(1 - m_2) b_1^h + m_1)}{f_1^h f_2^v} \right\} \\ &\quad + \frac{c^2}{2 f_1^h (1 - c^2 b_1^h b_2^v)^2} \left\{ -\frac{c(b_2^v)^v}{f_2^v} (\text{grad} f_1(b_1))^h + \frac{b_1^h}{f_1^h} (\text{grad} f_2(b_2))^v \right\}. \end{aligned}$$

Corollary 2. *If f_1 and f_2 are two harmonic functions, then f_1^h (resp. f_2^v) is harmonic if and only if*

$$\begin{aligned} \frac{b_1^h (c(1 - m_1) b_2^v + m_2)}{f_1^h f_2^v} + \frac{c^2}{2(1 - c^2 b_1^h b_2^v)} \left\{ \frac{b_2^v}{f_2^v} (\text{grad} f_1(b_1))^h - \frac{c(b_1^h)^2}{f_1^h} (\text{grad} f_2(b_2))^v \right\} &= 0, \\ \left(\text{resp. } \frac{b_2^v (c(1 - m_2) b_1^h + m_1)}{f_1^h f_2^v} + \frac{c^2}{2(1 - c^2 b_1^h b_2^v)} \left\{ \frac{b_1^h}{f_1^h} (\text{grad} f_2(b_2))^v - \frac{c(b_2^v)^v}{f_2^v} (\text{grad} f_1(b_1))^h \right\} \right) &= 0. \end{aligned}$$

Proof. As a direct consequence of Theorem 3.4. \square

Remark 4. 1/ If for all $i \in \{1, 2\}$, the gradient of f_i is parallel with respect to $\overset{i}{\nabla}$, then

$$\Delta(f_1^h) = \frac{(1 - m_1)cb_1^hb_2^v + m_2b_1^h}{f_1^h(f_2^v)^2(1 - c^2b_1^hb_2^v)} \text{ and } \Delta(f_2^v) = \frac{(1 - m_2)cb_1^hb_2^v + m_1b_2^v}{(f_1^h)^2f_2^v(1 - c^2b_1^hb_2^v)}.$$

2/ If $\varphi_i \in C^\infty(M_i)$ ($i = 1, 2$), then it is easy to calculate the $\Delta(\varphi_1^h)$ and $\Delta(\varphi_2^v)$ by using of course Lemmas 3.5, 3.6 and 3.7.

3/ We can calculate also, the bi-Laplacian of φ_1^h and φ_2^v .

4. Other remarkable metric tensor on a product manifold. Let c be an arbitrary real number and let g_i , ($i = 1, 2$) be a Riemannian metric tensors on M_i . Given a smooth positive function f_i on M_i , we define a metric tensor field on $M_1 \times M_2$ by

$$h_{f_1, f_2} = \pi_1^*g_1 + (f_1^h)^2\pi_2^*g_2 + \frac{c^2}{2}(f_2^v)^2df_1^h \odot df_1^h. \quad (39)$$

Where π_i , ($i = 1, 2$) is the projection of $M_1 \times M_2$ onto M_i .

It is the unique metric tensor such that for any $X_i, Y_i \in \Gamma(TM_i)$, ($i = 1, 2$)

$$\begin{cases} h_{f_1, f_2}(X_1^h, Y_1^h) = g_1(X_1, Y_1)^h + c^2(f_2^v)^2X_1(f_1)^hY_1(f_1)^h \\ h_{f_1, f_2}(X_2^v, Y_2^v) = (f_1^h)^2g_2(X_2, Y_2)^v \\ h_{f_1, f_2}(X_1^h, Y_2^v) = h_{f_1, f_2}(X_2^v, Y_1^h) = 0 \end{cases} \quad (40)$$

4.1. The Levi-Civita Connection.

Lemma 4.1. *Let (M_i, g_i) , ($i = 1, 2$) be a Riemannian manifold. The gradient of the lifts φ_1^h of φ_1 and φ_2^v of φ_2 to $M_1 \times_{f_1, f_2} M_2$ w.r.t. h_{f_1, f_2} are*

$$\text{grad}(\varphi_1^h) = (\text{grad}\varphi_1)^h - \frac{c^2(f_2^v)^2(\text{grad}\varphi(f_1))^h}{1 + c^2(f_2^v)^2b_1^h}(\text{grad}f_1)^h, \quad (41)$$

$$\text{grad}(\varphi_2^v) = \frac{1}{(f_1^h)^2}(\text{grad}\varphi_2)^v, \quad (42)$$

where $b_1 = \|\text{grad}f_1\|^2$.

Proof. It suffices to observe that

$$Z_1(f_1)^h = h_{f_1, f_2}\left(\frac{1}{1 + c^2(f_2^v)^2b_1^h}(\text{grad}f_1)^h, Z_1^h\right),$$

and so,

$$Z_1(\varphi_1)^h = h_{f_1, f_2}\left((\text{grad}\varphi_1)^h - \frac{c^2(f_2^v)^2(\text{grad}\varphi(f_1))^h}{1 + c^2(f_2^v)^2b_1^h}(\text{grad}f_1)^h, Z_1^h\right).$$

Therefor, the result follows by Equation (40) and Lemma 2.1. \square

let us compute the Levi-Civita connection of $M_1 \times_{f_1, f_2} M_2$ associated with the metric h_{f_1, f_2} in terms of the Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ associated with the metrics g_1 and g_2 respectively.

Proposition 4. *Let (M_i, g_i) , ($i = 1, 2$) be a Riemannian manifold. Then we have*

$$\begin{aligned} \nabla_{X_1^h}Y_1^h &= \overset{1}{\nabla}_{X_1}Y_1^h + \frac{(cf_2^v)^2H^{f_1}(X_1, Y_1)^h}{1 + (cf_2^v)^2b_1^h}(\text{grad}f_1)^h \\ &\quad - c^2f_2^v(X_1(\ln f_1)Y_1(\ln f_1))^h(\text{grad}f_2)^v \end{aligned} \quad (43)$$

$$\nabla_{X_2^v} Y_2^v = (\nabla_{X_2} Y_2)^v - \frac{f_1^h g_2(X_2, Y_2)^v}{1 + c^2 (f_2^v)^2 b_1^h} (grad f_1)^h \quad (44)$$

$$\nabla_{X_1^h} Y_2^v = \nabla_{Y_2^v} X_1^h = \frac{c^2 f_2^v Y_2(f_2)^v X_1(f_1)^h}{(1 + c^2 (f_2^v)^2 b_1^h)} (grad f_1)^h + (X_1(\ln f_1))^h Y_2^v, \quad (45)$$

Where H^{f_1} is the Hessian of f_1 .

Proof. It follows directly from Koszul formula and Equation (40). \square

4.2. The Laplacian of the lifts to M_1 and M_2 .

Theorem 4.2. *On a generalized warped product $(M_1 \times_{f_1 f_2} M_2, h_{f_1 f_2})$ with $m_1 = \dim M_1$ and $m_2 = \dim M_2$, Let $\varphi_1 : M_1 \rightarrow \mathbb{R}$ and $\varphi_2 : M_2 \rightarrow \mathbb{R}$ be a smooth functions. Then the Laplacian of the horizontal lift $\varphi_1 \circ \pi_1$ of φ_1 (resp. vertical lift $\varphi_2 \circ \pi_2$ of φ_2) to $M_1 \times_{f_1 f_2} M_2$ is given by*

$$\Delta(\varphi_1^h) = \Delta(\varphi_1)^h + \frac{m_2 (grad f_1(\varphi_1))^h}{f_1^h (1 + (c f_2^v)^2 b_1^h)} - \frac{(c f_2^v)^2}{1 + (c f_2^v)^2 b_1^h} \left\{ (grad f_1(\varphi_1))^h \Delta(f_1)^h \right. \\ \left. + H^{\varphi_1}(grad f_1, grad f_1)^h - \frac{(c f_2^v)^2 (grad f_1(\varphi_1))^h}{1 + (c f_2^v)^2 b_1^h} H^{f_1}(grad f_1, grad f_1)^h \right\} \quad (46)$$

$$\Delta(\varphi_2^v) = \frac{1}{(f_1^h)^2} \left\{ \Delta(\varphi_2)^v + \frac{c^2 f_2^v b_1^h (grad f_2(\varphi_2))^v}{1 + (c f_2^v)^2 b_1^h} \right\}. \quad (47)$$

Where $b_1 = \|grad f_1\|^2$.

Lemma 4.3. *On $(M_1 \times_{f_1 f_2} M_2, h_{f_1 f_2})$, if $\{e_1, \dots, e_{m_1}\}$ is the local frame field with respect to the metric g_1 and $\{e_{m_1+1}, \dots, e_{m_1+m_2}\}$ is the local frame field with respect to the metric g_2 , then $\{u_1, \dots, u_{m_1}, u_{m_1+1}, \dots, u_{m_1+m_2}\}$ is the local frame field with respect to the metric $h_{f_1 f_2}$, where*

$$u'_i = \begin{cases} -\frac{(c f_2^v)^2 a_i^h}{1 + (c f_2^v)^2 B_i^h} T_i^h + e_i^h, & i \in \{1, \dots, m_1\}; \\ \frac{1}{f_1^h} e_i^v, & i \in \{m_1 + 1, \dots, m_1 + m_2\}. \end{cases} \quad (48)$$

And for $i \in \{1, \dots, m_1\}$,

$$u_i = \frac{1}{\|u'_i\|} u'_i, \quad \|u'_i\|^2 = \frac{1 + (c f_2^v)^2 B_{i+1}^h}{1 + (c f_2^v)^2 B_i^h}, \quad B_i = \sum_{j=1}^{i-1} a_j^2, \quad T_i = \sum_{j=1}^{i-1} a_j e_j, \quad a_i = e_i(f_1).$$

Proof. For the proof of the lemma it is actually almost the most interesting result because it provides an algorithm for constructing $\{u_1, \dots, u_{m_1}, u_{m_1+1}, \dots, u_{m_1+m_2}\}$ from the family $\{e_1, \dots, e_{m_1}\}$ et $\{e_{m_1+1}, \dots, e_{m_1+m_2}\}$.

To do so, we use a limited recurrence (The Gram schmidt process)(see Lemma 3.5). \square

Remark 5. With the notations above, we have

- 1) T_1 is the zero vector field on M_1 , B_1 is the zero function on M_1 and A_{m_1+1} is the care of the gradient of f_1 .
- 2) For any $i \in \{1, \dots, m_1 + 1\}$

$$\begin{cases} T_i(f_1) = A_i = g(T_i, T_i), \\ u'_i(f_1^h) = \frac{a_i^h}{1 + (c f_2^v)^2 B_i^h}. \end{cases} \quad (49)$$

Lemma 4.4. *With the notations above, we have, for all $j \in \{1, \dots, m_1\}$*

$$(cf_2^v)^4(a_j^h)^2 \left(\sum_{i=j+1}^{m_1} \frac{(a_i^h)^2}{(1 + (cf_2^v)^2 B_i^h)(1 + (cf_2^v)^2 B_{i+1}^h)} \right) + \frac{1 + (cf_2^v)^2 B_j^h}{1 + (cf_2^v)^2 B_{j+1}^h} = 1 - \frac{(cf_2^v a_j^h)^2}{1 + (cf_2^v)^2 b_1^h}, \quad (50)$$

$$(cf_2^v)^2 \left(\sum_{i=j+1}^{m_1} \frac{(a_i^h)^2}{(1 + (cf_2^v)^2 B_i^h)(1 + (cf_2^v)^2 B_{i+1}^h)} \right) - \frac{1}{1 + (cf_2^v)^2 B_{j+1}^h} = \frac{-1}{1 + (cf_2^v)^2 b_1^h} \quad (51)$$

Proof. The proof is the following partial analogue of Lemma 3.7. \square

Proof of Theorem 4.2

The proof of the theorem is a very long calculation. that will be omitted here.

Now, we calculate the Laplacian of the lifts φ_1^h of φ_1 , using Lemma 4.3.

$$\begin{aligned} \Delta(\varphi_1^h) &= \sum_{j=1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{u_j} \text{grad}(\varphi_1^h), u_j) \\ &= \sum_{j=1}^{m_1} h_{f_1 f_2}(\nabla_{u_j} \text{grad}(\varphi_1^h), u_j) + \sum_{j=m_1+1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{u_j} \text{grad}(\varphi_1^h), u_j) \\ &= \sum_{j=1}^{m_1} \frac{1}{\|u'\|^2} h_{f_1 f_2}(\nabla_{u'_j} \text{grad}(\varphi_1^h), u'_j) + \left(\frac{1}{f_1^h}\right)^2 \sum_{j=m_1+1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{e_j^v} \text{grad}(\varphi_1^h), e_j^v) \quad (52) \end{aligned}$$

Calculate, the seconde term on the right-hand side of the last equation above

$$\left(\frac{1}{f_1^h}\right)^2 \sum_{j=m_1+1}^{m_1+m_2} h_{f_1 f_2}(\nabla_{e_j^v} \text{grad}(\varphi_1^h), e_j^v) = \frac{m_2(\text{grad} f_1(\varphi_1))^h}{f_1^h(1 + (cf_2^v)^2 b_1^h)}.$$

Calculate, the first term on the right-hand side of Equation (52). Straightforward calculation using Lemmas 4.4 gives

$$\begin{aligned} \sum_{j=1}^{m_1} \frac{1}{\|u'\|^2} h_{f_1 f_2}(\nabla_{u'_j} \text{grad}(\varphi_1^h), u'_j) &= \sum_{j=1}^{m_1} h_{f_1 f_2}(\nabla_{e_j^h} \text{grad}(\varphi_1^h), e_j^h) \\ &\quad - \frac{(cf_2^v)^2}{1 + (cf_2^v)^2 b_1^h} \sum_{1 \leq i, j \leq m_1} a_i^h a_j^h h_{f_1 f_2}(\nabla_{e_j^h} \text{grad}(\varphi_1^h), e_j^h) \\ &= \sum_{j=1}^{m_1} h_{f_1 f_2}(\nabla_{e_j^h} \text{grad}(\varphi_1^h), e_j^h) - \frac{(cf_2^v)^2}{1 + (cf_2^v)^2 b_1^h} h_{f_1 f_2}(\nabla_{(\text{grad} f_1)^h} \text{grad}(\varphi_1^h), (\text{grad} f_1)^h). \end{aligned}$$

Using Proposition 4, we obtain Equation (46).

The seconde assertion is similar.

Corollary 3. *Let (M_i, g_i) ($i = 1, 2$) be a connected riemannian manifolds. If f_1 is a harmonic function such that $\text{grad} f_1 \neq 0$, then f_1^h is harmonic if and only if*

$$c \neq 0, \quad f_2 \text{ is a constant function and } H^{f_1}(\text{grad} f_1, \text{grad} f_1) = m_2 b_1 (b_1 + \frac{1}{c^2 f_2^2}),$$

If f_2 is harmonic function, then f_2^v is harmonic if and only if

$$c = 0 \text{ or } (f_1 \text{ or } f_2 \text{ is a constant function}).$$

Proof. As a direct consequence of Theorem 3.4. \square

4.3. The Curvature tensors. Let $\bar{\mathcal{R}}^i$ ($i = 1, 2$) and \mathcal{R} be the Riemannian curvature tensors with respect to g_i and $g_{f_1 f_2}$ respectively. In the following proposition, we express the curvature \mathcal{R} of the connection ∇ in terms of the warping functions f_1, f_2 and the curvatures $\bar{\mathcal{R}}^1$ and $\bar{\mathcal{R}}^2$ of $\bar{\nabla}^1$ and $\bar{\nabla}^2$ respectively.

Proposition 5. *Let (M_i, g_i) , ($i = 1, 2$) be a connected Riemannian manifold and let $f_1 \in C^\infty(M_1)$ be a non-constant positive function. Assume that the gradient of f_i is parallel with respect to $\bar{\nabla}^i$ ($i = 1, 2$). Then for any $X_i, Y_i, Z_i \in \Gamma(TM_i)$ ($i = 1, 2$) we have*

$$\begin{aligned} \mathcal{R}(X_1^h, Y_1^h)Z_1^h &= (\bar{\mathcal{R}}^1(X_1, Y_1)Z_1)^h, \\ \mathcal{R}(X_2^v, Y_2^v)Z_2^v &= (\bar{\mathcal{R}}^2(X_2, Y_2)Z_2)^v - \frac{b_1}{1+(cf_2^v)^2 b_1} \{(X_2 \wedge_{g_2} Y_2)Z_2\}^v \\ &\quad + \frac{c^2 f_1^h f_2^v b_1}{(1+(cf_2^v)^2 b_1)^2} \{((X_2 \wedge_{g_2} Y_2)Z_2)(f_2)\}^v (grad f_1)^h, \\ \mathcal{R}(X_1^h, Y_1^h)Z_2^v &= 0, \\ \mathcal{R}(X_2^v, Y_2^v)Z_1^h &= \frac{c^2 f_2^v b_1 (Z(f_1))^h}{f_1^h (1+(cf_2^v)^2 b_1)} \{(X_2 \wedge_{g_2} Y_2)grad f_2\}^v, \\ \mathcal{R}(X_1^h, Y_2^v)Z_1^h &= \frac{c^2 X_1(\ln f_1)^h Z_1(\ln f_1)^h Y_2(f_2)^v}{1+(cf_2^v)^2 b_1} (grad f_2)^v, \\ \mathcal{R}(X_1^h, Y_2^v)Z_2^v &= \frac{c^2 X_1(\ln f_1)^h}{1+(cf_2^v)^2 b_1} \left\{ f_2^v b_1 ((grad f_2 \wedge_{g_2} Y_2)Z_2)^v - \frac{f_1^h Y_2(f_2)^v Z_2(f_2)^v}{1+(cf_2^v)^2 b_1} (grad f_1)^h \right\}. \end{aligned}$$

where the wedge product $(X_2 \wedge_{g_2} Y_2)Z_2 = g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2$.

Proof. Long but straightforward computations using Proposition (4) and Lemma (4.1). \square

As direct consequence of Proposition 5 we obtain

Corollary 4. *Let (M_i, g_i) , ($i = 1, 2$) be a Riemannian manifold. Assume that the gradient of f_1 is parallel with respect to $\bar{\nabla}^1$. If $(M_1 \times M_2, h_{f_1 f_2})$ is flat then the base (M_1, g_1) is flat and the fiber (M_2, g_2) is space of constant sectional curvature $k = \frac{b_1}{1+(cf_2^v)^2 b_1}$.*

Now consider the Ricci curvature Ric of a generalized warped product, writing $(\text{Ric}_1)^h$ for the lift (pullback by π_1) of the Ricci curvature of M_1 , and similarly for $(\text{Ric}_2)^v$.

Proposition 6. *Under the same assumptions as in Proposition 5, let $\text{Ric}_1, \text{Ric}_2$ and Ric be the Ricci curvature tensors with respect to g_1, g_2 and $h_{f_1 f_2}$ respectively.*

let $X_1, Y_1 \in \Gamma(TM_1)$ and $X_2, Y_2 \in \Gamma(TM_2)$, then we have

$$\mathcal{R}ic(X_1^h, Y_1^h) = \mathcal{R}ic_1(X_1, Y_1)^h - \frac{c^2 b_2^v}{1+(cf_2^v)^2 b_1} X_1(\ln f_1)^h Y_1(\ln f_1)^h,$$

$$\mathcal{R}ic(X_1^h, Y_2^v) = \frac{c^2(m_2-1)b_1 f_2^v}{1+(cf_2^v)^2 b_1} X_1(\ln f_1)^h Y_2(f_2)^v,$$

$$\mathcal{R}ic(X_2^v, Y_2^v) = \mathcal{R}ic_2(X_2, Y_2)^v + \frac{c^2 b_1}{(1+(cf_2^v)^2 b_1)^2} X_2(f_2)^v Y_2(f_2)^v - \frac{(m_2-1)b_1}{1+(cf_2^v)^2 b_1} g_2(X_2, Y_2)^v.$$

Where $m_2 = \dim M_2$.

Proof. Long but straightforward computations using Propositions (4, 5) and Lemmas (4.1, 4.3 and 4.4). \square

Corollary 5. *Under the same assumptions as in Proposition 5, let $\mathcal{S}_1, \mathcal{S}_2$ and \mathcal{S} be the scalar curvature with respect to g_1, g_2 and $g_{f_1 f_2}$ respectively. Then the following equation holds*

$$\mathcal{S} = \mathcal{S}_1^h + \frac{1}{(f_1^h)^2} \mathcal{S}_2^v - \frac{m_2(m_2-1)b_1}{(f_1^h)^2(1+(cf_2^v)^2 b_1)}.$$

Proof. Follows from Propositions (4, 5) and Lemmas (4.1, 4.3 and 4.4). \square

Corollary 6. *Under the same assumptions as in Proposition 5, let (M_i, g_i) ($i = 1, 2$) be a riemannian manifold with constant sectional curvature k_i . Then*

$$\mathcal{S}(p_1, p_2) = m_1(m_1 - 1)k_1 + \frac{m_2(m_2 - 1)}{f_1(p_1)^2} \left(k_2 - \frac{\|grad f_1\|_{p_1}}{1 + (cf_2(p_2))^2 \|grad f_1\|_{p_1}} \right).$$

Proof. We know that if (M_i, g_i) ($i = 1, 2$) have constant sectional curvature k_i , then $\mathcal{S}_i(p_i) = m_i(m_i - 1)k_i$. By Corollary 5 follows. \square

Acknowledgement: A big part of this work was done at The Raphel Salem Laboratory of mathematics, University of Rouen (France), Rafik Nasri would like to thank Raynaud de Fitte Paul, Siman Raulot for very useful discussions and the Mathematic section for their hospitality.

REFERENCES

- [1] Y. Ait Amrane, R. Nasri and A. Zeglou, *Warped Poisson brackets on warped products*, J.Geom.Mecha., Amer. Inst. Math. Sci. 6, 3(2014), 279-296.
- [2] J. K. Beem, P. E. Ehrlich and Th. G. Powell, *Warped product manifolds in relativity*, North-Holland Publishing company, (1982), 41-56.
- [3] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., 145, (1969), 1-49.
- [4] R. Nasri and M. Djaa, *Sur la courbure des variétés riemanniennes produits*, Sciences et Technologie, 24, A(2006), 15-20.
- [5] R. Nasri and M.Djaa, *On the geometry of the product Riemannian manifold with the Poisson structure*, Int. Elect.J.Geom., 3, 2(2010) 1-14.
- [6] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New-York, 1983.
- [7] Ponge, R., Reckziegel, H, *Twisted products in pseudo-Riemannian geometry*, Geom. Dedic. 48, (1993), 15-25.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: rmag1math@yahoo.fr

E-mail address: djaamustapha@live.com